

Groups not acting on compact metric spaces by homeomorphisms

Azer Akhmedov

Abstract: We show that the direct sum of uncountably many non-Abelian groups does not embed into the group of homeomorphisms of a compact metric space.

The main result of this note does not seem to exist in the literature. Among other things, it provides very simple examples of left-orderable groups of continuum cardinality which do not embed in $\text{Homeo}_+(\mathbb{R})$ ¹

Theorem 1. Let X be a compact metric space, I be an uncountable set, and G_α be a non-Abelian group, for all $\alpha \in I$. Then the direct sum $\bigoplus_{\alpha \in I} G_\alpha$ does not embed into the group of homeomorphisms of X .

Proof. We will assume that the direct sum $\bigoplus_{\alpha \in I} G_\alpha$ embeds in $\text{Homeo}(X)$.

For all $\alpha \in I$, let f_α, g_α be some non-commuting elements in G_α . We will denote the metric on X by $d(\cdot, \cdot)$. Notice that the group $\text{Homeo}(X)$ of homeomorphisms of X also becomes a metric space where for $\phi, \psi \in \text{Homeo}(X)$, we define the metric by $D(\phi, \psi) = \sup_{x \in X} d(\phi(x), \psi(x))$.

By compactness of X , we have a sequence $C_n, n \geq 1$ of finite subsets of X such that

- (i) for all $n \geq 1$, C_n is a $\frac{1}{n}$ -net, i.e. for all $x \in X$, there exists $z \in C_n$ such that $d(z, x) < \frac{1}{n}$,
- (ii) $C_n \subseteq C_{n+1}$, for all $n \geq 1$,
- (iii) $\bigcup_{n \geq 1} C_n = X$.

Let $C_n = \{x_1^{(n)}, \dots, x_{p_n}^{(n)}\}, n \geq 1$. For all $n \geq 1$ and $\delta > 0$, we will also write

$$\mathcal{F}_{n,\delta} = \{\phi \in \text{Homeo}(X) \mid \text{diam} \phi(B_{\frac{\delta}{2}}(x_i^{(n)})) < \delta, \forall i \in \{1, \dots, p_n\}\}$$

¹It is well known that a countable group is left-orderable iff it embeds into $\text{Homeo}_+(\mathbb{R})$. Thus it becomes an interesting question whether or not there exists a left-orderable group of continuum cardinality which does not embed in $\text{Homeo}_+(\mathbb{R})$. This question has been addressed in a very recent paper [M] where two different examples have been constructed.

Since $\text{Homeo}(X) \setminus \{1\} = \bigcup_{i \geq 1} \{\phi \mid D(\phi, 1) > \frac{1}{i}\}$ there exists $c > 0$ and an uncountable set $I_1 \subset I$ such that for all $\alpha \in I_1$ we have $D([f_\alpha, g_\alpha], 1) > c$.

By compactness, all homeomorphisms of X are uniformly continuous. Then there exists an uncountable set $I_2 \subset I_1$ and positive integers n, m such that $\max\{\frac{1}{n}, \frac{1}{m}\} < \frac{c}{10}$ and for all $\beta \in I_2$ we have $\{f_\beta, g_\beta\} \subseteq \mathcal{F}_{n, \frac{1}{m}}$.

Then, since $\text{Homeo}(X)$ is separable, there exists an uncountable set $I_3 \subset I_2$ and $f_\star, g_\star \in \text{Homeo}(X)$ such that for all $\gamma \in I_3$ we have $\max\{D(f_\gamma, f_\star), D(g_\gamma, g_\star)\} < \frac{1}{100n}$.

Then for all $\gamma, \eta \in I_3$ and for all $x \in X$ we have the inequalities

$$d(g_\eta f_\gamma(x), g_\gamma f_\gamma(x)) < \frac{1}{50n} \text{ and } d(f_\gamma g_\eta(x), f_\gamma g_\gamma(x)) < \frac{c}{5}.$$

On the other hand, for any two distinct $\eta, \gamma \in I_3$ there exists $x_0 \in X$ such that $d(f_\gamma g_\gamma(x_0), g_\gamma f_\gamma(x_0)) > c$.

By triangle inequality, we obtain that $d(g_\eta f_\gamma(x_0), f_\gamma g_\eta(x_0)) > \frac{c}{2}$. Thus $[g_\eta, f_\gamma] \neq 1$. Contradiction. \square

Remark. To obtain interesting applications of the main theorem, X can be taken to be an arbitrary compact manifold (with boundary). In the case of a closed interval $I \cong [0, 1]$, we obtain a non-embeddability result into $\text{Homeo}_+(\mathbb{R})$, i.e. the direct sum of uncountably many isomorphic copies of a non-Abelian group G does not embed in $\text{Homeo}_+(\mathbb{R})$. The result of Theorem 1 can be extended to a much larger class of topological spaces. One can also drop or replace the non-Abelian condition on the group G in certain special contexts.

References:

[M] K.Mann, Left-orderable groups that don't act on the line. *Mathematische Zeitschrift* **vol. 280**, issue 3, (2015) 905-918.

AZER AKHMEDOV, DEPARTMENT OF MATHEMATICS, NORTH DAKOTA STATE UNIVERSITY, FARGO, ND, 58108, USA

E-mail address: `azer.akhmedov@ndsu.edu`